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A Toeplitz-Like Operator with Rational Matrix Symbol Having Poles on the Unit Circle: Fredholm Properties

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Abstract

This paper concerns the analysis of an unbounded Toeplitz-like operator generated by a rational matrix function having poles on the unit circle \mathbb{T} . It extends the analysis of such operators generated by scalar rational functions with poles on \mathbb{T} found in Groenewald et al. (Oper Theory Adv Appl 271:239–268, 2018; Oper Theory Adv Appl 272:133–154, 2019; Integr Equ Oper Theory 91, 2019). A Wiener–Hopf type factorization of rational matrix functions with poles and zeroes on \mathbb{T} is proved and then used to analyze the Fredholm properties of such Toeplitz-like operators. A formula for the index, based on the factorization, is given. Furthermore, it is shown that the determinant of the matrix function having no zeroes on \mathbb{T} is not sufficient for the Toeplitz-like operator to be Fredholm, in contrast to the classical case.

Keywords Toeplitz operators · Unbounded operators · Fredholm properties · Rational matrix functions · Wiener–Hopf factorization

Mathematics Subject Classification Primary 47B35 · 47A53; Secondary 47A68

1 Introduction

This paper is a continuation of [11–13], where Toeplitz-like operators with rational symbols with poles on the unit circle \mathbb{T} were studied. Whilst the aim of [11–13] was to

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Communicated by Seppo Hassi.

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analyze such Toeplitz-like operators with scalar symbols, in this paper we will focus on such Toeplitz-like operators with matrix symbol.

First we introduce some notation. For positive integers m and n , let $\text{Rat}^{m \times n}$ denote the space of $m \times n$ rational matrix functions, abbreviated to Rat^m if $n = 1$. We write $\text{Rat}^{m \times n}(\mathbb{T})$ for the functions in $\text{Rat}^{m \times n}$ with poles only on \mathbb{T} and with $\text{Rat}_0^{m \times n}(\mathbb{T})$ we indicate the functions in $\text{Rat}^{m \times n}$ that are strictly proper, that is, whose limit at ∞ exists and is equal to the zero-matrix. Also here, if $n = 1$ we write $\text{Rat}^m(\mathbb{T})$ and $\text{Rat}_0^m(\mathbb{T})$ instead of $\text{Rat}^{m \times 1}(\mathbb{T})$ and $\text{Rat}_0^{m \times 1}(\mathbb{T})$, respectively. In the scalar case, i.e., $m = n = 1$, we simply write Rat , $\text{Rat}(\mathbb{T})$ and $\text{Rat}_0(\mathbb{T})$, as was done in [11–13].

A rational matrix function has a pole at z if any of its entries has a pole at z . A zero of a square rational matrix function Ω is a pole of its inverse $\Omega^{-1}(z) = \Omega(z)^{-1}$, see for example [3]. In the case of a square matrix polynomial P , the zeroes coincide with the zeroes of the polynomial $\det P(z)$, but for square rational matrix functions, zeroes can also occur at points where the determinant is nonzero.

The space of $m \times n$ matrix polynomials will be denoted by $\mathcal{P}^{m \times n}$, abbreviated to \mathcal{P}^m if $n = 1$ and to \mathcal{P} if $m = n = 1$. For all positive integers k we write $\mathcal{P}_k^{m \times n}$, \mathcal{P}_k^m and \mathcal{P}_k for the polynomials in $\mathcal{P}^{m \times n}$, \mathcal{P}^m and \mathcal{P} of degree at most k . By L_m^p and H_m^p we shall mean vector-functions with m -components who are all in L^p and H^p , respectively.

We can now define our Toeplitz-like operators. Let $\Omega \in \text{Rat}^{m \times m}$ with possibly poles on \mathbb{T} and $\det \Omega(z) \neq 0$. Define $T_\Omega (H_m^p \rightarrow H_m^p)$, $1 < p < \infty$, by

$$\text{Dom}(T_\Omega) = \left\{ f \in H_m^p : \begin{array}{l} \Omega f = h + r \text{ where } h \in L_m^p(\mathbb{T}), \\ \text{and } r \in \text{Rat}_0^m(\mathbb{T}) \end{array} \right\}, \quad (1.1)$$

$$T_\Omega f = \mathbb{P}h \text{ with } \mathbb{P} \text{ the Riesz projection of } L_m^p(\mathbb{T}) \text{ onto } H_m^p.$$

By the Riesz projection, \mathbb{P} , we mean the projection of L_m^p onto H_m^p due to M. Riesz, see for example pages 149–153 in [14], in contrast to the Riesz projection in spectral operator theory, due to F. Riesz, see for example pages 9–13 in [6]. For simplicity we will consider the square case only, but many of the results in this paper extend to the non-square case, i.e., $m \neq n$.

The aim of this paper is the determination of Fredholm properties of the Toeplitz-like operator T_Ω defined above. For the scalar case, the Fredholm-properties of Toeplitz-like operators were studied in [11]. For the classical case, i.e., without poles on the unit circle, the Fredholm-properties appear in Chapters XXIII and XXIV of [7]. In that case, the block Toeplitz operator is Fredholm exactly when the determinant of the symbol has no zeroes on \mathbb{T} (Theorem XXIII.4.3 in [7]). As we will see later, see Sect. 7, this is not the case when poles on the unit circle are allowed, due to possible pole-zero cancellation.

An essential ingredient in the analysis of Fredholm properties of Toeplitz operators with matrix symbols is played by Wiener–Hopf factorization; cf., Theorem XXIV.3.1 of [7]. This allows one to determine invertibility conditions and Fredholm-properties of the block Toeplitz operator; cf., Theorem XXIV.4.1 and Theorem XXIV.4.2 of [7]. Here too we base our analysis on a Wiener–Hopf-type factorization.

Main Results In our first main result, using an adaptation of the construction in [2] we prove a Wiener–Hopf type factorization for a rational matrix function with poles on the unit circle. We call an $m \times m$ rational matrix function a plus function if it has no poles in the closed unit disk $\overline{\mathbb{D}}$ and we call it a minus function if it has no poles outside the open unit disk \mathbb{D} , with the point at infinity included.

Theorem 1.1 *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$. Then*

$$\Omega(z) = z^{-k} \Omega_{-}(z) \Omega_{\circ}(z) P_0(z) \Omega_{+}(z), \quad (1.2)$$

for some $k \geq 0$, $\Omega_{+}, \Omega_{\circ}, \Omega_{-} \in \text{Rat}^{m \times m}$ and $P_0 \in \mathcal{P}^{m \times m}$ such that Ω_{-} and $(\Omega_{-})^{-1}$ are minus functions, Ω_{+} and $(\Omega_{+})^{-1}$ are plus functions, $\Omega_{\circ} = \text{Diag}_{j=1}^m(\phi_j)$ with each $\phi_j \in \text{Rat}(\mathbb{T})$ having zeroes and poles only on \mathbb{T} , and P_0 is a lower triangular matrix polynomial with $\det(P_0(z)) = z^N$ for some integer $N \geq 0$.

Note that, the diagonal entries of P_0 must be of the form z^{n_1}, \dots, z^{n_m} . However, since P_0 is assumed to be lower triangular and not diagonal, it is not possible to order them (increasing or decreasing in power of z) without disrupting the lower triangular structure. This is in sharp contrast to the classical Wiener–Hopf factorization result where the entries on the diagonal can be ordered to have increasing degrees and subject to this ordering become unique, see for example [2,9].

In the case studied here, with poles on \mathbb{T} allowed, we do not find a very satisfactory uniqueness claim. As in the classical case, the plus function Ω_{+} and Ω_{-} are, in general, not unique. However, also the lower triangular matrix polynomial P_0 appears to be far from unique (as shown in Examples 7.2 and 7.3 below). Subject to the restriction of a special form of the factorization (1.2), which can always be obtained, it is possible to prove that the factor Ω_{\circ} is unique.

Theorem 1.2 *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$. Let $q(z)$ be the least common multiple of all denominators of the entries of Ω , and write $q(z) = q_{-}(z) q_{\circ}(z) q_{+}(z)$, where q_{+} has zeroes only outside $\overline{\mathbb{D}}$, q_{-} has zeroes only inside \mathbb{D} , and q_{\circ} has zeroes only on \mathbb{T} . Then Ω admits a factorization (1.2) as in Theorem 1.1 with the additional properties that*

$$\begin{aligned} P_{+}(z) &:= q_{+}(z) \Omega_{+}(z), \quad D_{\circ}(z) := q_{\circ}(z) \Omega_{\circ}(z), \\ P_{-}\left(\frac{1}{z}\right) &:= z^{-\deg q_{-}} q_{-}(z) \Omega_{-}(z), \end{aligned} \quad (1.3)$$

define matrix polynomials P_{+} and P_{-} with no roots inside $\overline{\mathbb{D}}$, D_{\circ} is a diagonal polynomial, $D_{\circ}(z) = \text{Diag}_{j=1}^m(p_j^{\circ}(z))$ with p_{j+1}° dividing p_j° for $j = 1, \dots, m-1$, k is the smallest number so that P_0 is still a polynomial, i.e., if $k > 0$ then $P_0(0) \neq 0$, and the off-diagonal entries in P_0 have a lower degree than the diagonal entry in the same row. Among all factorizations (1.2) of Ω satisfying these additional conditions, Ω_{\circ} is uniquely determined.

In fact, as observed in Corollary 4.1 below, the diagonal matrix D_{\circ} corresponds to what we define in Sect. 3 as the Smith form of the polynomial $P_1(z) := q(z) \Omega(z)$ with respect to \mathbb{T} , and can be directly determined by the classical Smith form of P_1 .

In Example 7.3 below, for the case of 2×2 matrix function, subject to an additional constraint it is possible to diagonalize the factor $\Omega_o(z)P_0(z)$ in 1.2. Whether it is possible to diagonalize $\Omega_o(z)P_0(z)$ in general remains unclear, we have proof nor counterexample, even for the 2×2 case. However, the arguments used in the proofs of the classical cases, without poles on \mathbb{T} , do not appear to generalize easily to the case considered in the present paper.

As in the case of scalar rational functions, the Wiener–Hopf type factorization of Theorem 1.1 allows us to factorize the Toeplitz-like operator along the matrix function factorization.

Theorem 1.3 *Let $\Omega(z) = z^{-k}\Omega_-(z)\Omega_o(z)P_0(z)\Omega_+(z)$ be a Wiener–Hopf type factorization of Ω as in Theorem 1.1. Then*

$$T_\Omega = T_{\Omega_-}T_{z^{-k}}T_{\Omega_o}T_{P_0}T_{\Omega_+}.$$

Our next, and final, main result uses the factorization of T_Ω to characterize the Fredholmness of T_Ω and determine the Fredholm index of T_Ω in terms of properties of the functions of the Wiener–Hopf type factorization of Ω in case T_Ω is Fredholm.

Theorem 1.4 *Let $\Omega(z) = z^{-k}\Omega_-(z)\Omega_o(z)P_0(z)\Omega_+(z)$ be the Wiener–Hopf type factorization of Ω as in Theorem 1.1 with $\Omega_o(z) = \text{Diag}_{j=1}^n(s_j(z)/q_j(z))$ with $s_j, q_j \in \mathcal{P}$ co-prime with roots only on \mathbb{T} and z^{n_j} the j th diagonal entry of P_0 , for $j = 1, \dots, m$. Then T_Ω is Fredholm if and only if T_{Ω_o} is Fredholm, which happens exactly when all s_1, \dots, s_m are constant. In case T_Ω is Fredholm we have*

$$\begin{aligned} \text{Index}(T_\Omega) &= mk + \text{Index}(T_{\Omega_o}) + \text{Index}(T_{P_0}) \\ &= mk + \sum_{j=1}^m \deg q_j - \sum_{j=1}^m n_j. \end{aligned}$$

Comparison with the Literature There are not many known results concerning unbounded (or closed) Toeplitz operators with matrix symbols. We use a factorization based on the Smith form for matrix polynomials (Theorem 1 and Theorem 2 of Gantmacher, Chapter VI [4]) which resembles the Wiener–Hopf factorization for rational matrix functions without poles on the unit circle appearing in Theorem 2.1 of [2]. The proof of Theorem 2.1 of [2] uses the Smith form as a starting point.

Factorization of matrix functions, however, has a long tradition. Wiener–Hopf type factorizations of matrix functions relative to a given contour (with certain properties, but not necessarily equal to \mathbb{T}) on which they can have no poles, appeared, for example, in Gohberg and Krein [9] and Clancey and Gohberg [2]; factorization of matrix functions as solutions to barrier problems in complex function theory appear as early as 1908 in Plemelj [16], whereas factorization of rational matrix functions relative to the unit circle appears as spectral factorization in electrical engineering in Belevitch [1] and Youla [18]. In all of these cases, however, there is the restriction that the (matrix) functions can have no poles on the contour.

The proof of the Wiener–Hopf type factorization for a rational matrix function with poles on the unit circle of Theorem 1.1 is based on the approach found in [2]. However,

there is a slight oversight in the proof of Theorem 2.1 in Chapter 1 of [2]. An application of Lemma 3.4 below would eliminate this. It is not true that $LD(z)P(z) = D(z)LP(z)$ where L is a lower triangular elementary matrix with ones on the main diagonal and only one row of nonzero entries off the main diagonal, $D(z) = \text{Diag}(z^{n_j})$ is a diagonal matrix with $n_1 \geq n_2 \geq \dots \geq n_m$ and $P(z)$ a matrix polynomial with $\det P$ having a zero only at $z = 0$. Applying Lemma 3.4 we can write $D(z)LP(z) = G(z)D(z)P(z)$ where $G(z)$ is a lower triangular minus matrix function with ones on the main diagonal. The result of Theorem 2.1 in Chapter 1 of [2] follows using this adaptation.

There is some divergence from the situation where Ω has no poles on \mathbb{T} . In that case, from Theorem XXIV.4.3 in [7] we have that T_Ω is Fredholm if and only if $\det \Omega(z) \neq 0$ for $z \in \mathbb{T}$. When poles on \mathbb{T} are allowed there could be pole-zero cancellation in the determinant of $\Omega(z)$ for $z \in \mathbb{T}$, for example $\Omega(z) = \text{Diag}\left(\frac{z+1}{z-1}, \frac{z-1}{z+1}\right)$ has $\det \Omega(z) \equiv 1$. However, T_Ω is not Fredholm. Thus, there are cases where $\det \Omega(z) \neq 0$ for $z \in \mathbb{T}$ but T_Ω is not Fredholm, which does not happen in the case where Ω has no poles on \mathbb{T} .

Overview The paper is organized as follows: Besides the current introduction, the paper consists of six sections. In Sect. 2 we prove basic results concerning the Toeplitz-like operator T_Ω . In the following section, Sect. 3, we derive various factorization results required for the proof of the Wiener–Hopf type factorization of Theorem 1.1. The proofs of Theorems 1.1 and 1.2 will be given in Sect. 4, followed by an example that illustrates the construction of the Wiener–Hopf type factorization in Sect. 5. In the next section, we prove the factorization of the Toeplitz-like operator, i.e., Theorem 1.3. Finally, in Sect. 7 we consider the Fredholm properties of T_Ω , including a proof of Theorem 1.4, and we present some examples that exhibit the non-uniqueness in our Wiener–Hopf type factorization.

2 Basic Properties of T_Ω

Using similar arguments as in the scalar case, see [11], we determine various basic properties of the Toeplitz-like operator T_Ω . Some of these results can be derived by restricting to the entries of Ω , in which case we give minimal details of the proof. We start with an analogue of Proposition 2.1 of [11].

Proposition 2.1 *Let $\Omega \in \text{Rat}^{m \times m}$, possibly with poles on \mathbb{T} . Then T_Ω is a well-defined, closed, densely defined linear operator on H_m^p . More specifically, $\mathcal{P}^m \subset \text{Dom}(T_\Omega)$. Moreover, $\text{Dom}(T_\Omega)$ is invariant under the forward shift $S_+ = T_{zI_m}$ on H_m^p and we have*

$$S_- T_\Omega S_+ f = T_\Omega f \quad \text{for all } f \in \text{Dom}(T_\Omega), \quad (2.1)$$

where $S_- = T_{z^{-1}I_m}$ on H_m^p .

We first give two lemmas, without proof, which can be derived in a way analogous to the scalar case, starting with the analogue of Lemma 2.4 in [11].

Lemma 2.2 *Given $\Omega \in \text{Rat}^{m \times m}$, we can write $\Omega(z) = \Omega_1(z) + \Omega_2(z)$, where $\Omega_1 \in \text{Rat}^{m \times m}$ with no poles on \mathbb{T} and $\Omega_2 \in \text{Rat}_0^{m \times m}(\mathbb{T})$.*

The above lemma allows one to reduce certain questions to the case where $\Omega \in \text{Rat}^{m \times m}(\mathbb{T})$. In that case, with arguments similar to the ones used in the proof of Lemma 2.3 in [11], the domain can be described as in the next lemma.

Lemma 2.3 *Let $\Omega \in \text{Rat}^{m \times m}(\mathbb{T})$. Write $\Omega = q(z)^{-1}P(z)$ with $P \in \mathcal{P}^{m \times m}$ and $q \in \mathcal{P}$, q having roots only on \mathbb{T} . Then*

$$\text{Dom}(T_\Omega) = \left\{ g \in H_m^p : \Omega g = h + q^{-1}r, \text{ with } h \in H_m^p, r \in \mathcal{P}_{\deg(q)-1}^m \right\},$$

and $T_\omega g = h$ for $g \in \text{Dom}(T_\Omega)$.

Sketch of the proof of Proposition 2.1 The proof mostly follows by direct generalization of the arguments in [11, 12], sometimes reducing to results for $T_{\omega_{ij}}$, where $\Omega = [\omega_{ij}]_{i,j=1}^m$.

For $\rho \in \text{Rat}_0^m(\mathbb{T})$, using a similar argument as in Lemma 2.2 in [11] on its entries, it follows that ρ is identically zero whenever $\rho \in L_m^p$. Now following the argument in Proposition 2.1 in [11] one obtains that T_Ω is well-defined.

Let $\Omega \in \text{Rat}^{m \times m}$. By Lemma 2.2, we can write $\Omega(z) = \Omega_1(z) + \Omega_2(z)$, where $\Omega_1 \in \text{Rat}^{m \times m}$ with no poles on \mathbb{T} and $\Omega_2 \in \text{Rat}_0^{m \times m}(\mathbb{T})$. Then $T_\Omega = T_{\Omega_1} + T_{\Omega_2}$ and the domains of T_Ω and T_{Ω_2} coincide. To see this, note that $f \in \text{Dom}(T_\Omega)$ if and only if $f \in \text{Dom}(T_{\Omega_2})$ and that the latter is the case if and only if $\Omega_2 f = h_2 + \rho$ where $h_2 \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$. Now for such a function f consider $\Omega f = \Omega_1 f + \Omega_2 f$. Since $\Omega_1 \in L_\infty^{m \times m}$, also $\Omega_1 f \in L_m^p$. Moreover, we have

$$\Omega f = \Omega_1 f + \Omega_2 f = (\Omega_1 f + h_2) + \rho = h + \rho,$$

where $h = \Omega_1 f + h_2$. Now

$$\mathbb{P}h = \mathbb{P}(\Omega_1 f) + \mathbb{P}h = T_{\Omega_1} f + T_{\Omega_2} f = T_\Omega f$$

as desired. Hence, for various qualitative properties of T_Ω , including closedness, we may assume without loss of generality that $\Omega \in \text{Rat}_0^{m \times m}(\mathbb{T})$.

Assume $\Omega \in \text{Rat}_0^{m \times m}(\mathbb{T})$. Then we can write $\Omega(z) = q(z)^{-1}P(z)$ where $q \in \mathcal{P}_\ell$ has zeroes only on \mathbb{T} and $P \in \mathcal{P}_{\ell-1}^{m \times m}$ for some $\ell \in \mathbb{N}$. Using a similar argument as in the proof of Lemma 2.3 in [11] we can show that $f \in \text{Dom}(T_\Omega)$ if and only if $\Omega f = h + q^{-1}r$, where $h \in H_m^p$ and $r \in \mathcal{P}_{\ell-1}^m$. Moreover, r and h are unique, and in that case $T_\Omega f = h$. Now using a similar argument as in the proof of Proposition 2.1 in [11] it follows that T_Ω is closed and that the domain of T_Ω contains all the polynomials and so T_Ω is densely defined.

Finally, to prove $\text{Dom}(T_\omega)$ is invariant under S_+ as well as (2.1), let $f \in \text{Dom}(T_\Omega)$. Then $\Omega f = h + \rho$ for $h \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$. Then $\Omega S_+ f = \Omega z f = zh + z\rho$. Apply the Euclidean algorithm entrywise to write $z\rho = \rho' + c$ with $\rho' \in \text{Rat}_0^m(\mathbb{T})$, with the same poles as ρ , and $c \in \mathbb{C}^m$. Hence $\Omega S_+ f = (zh + c) + \rho' \in L_m^p + \text{Rat}_0^m(\mathbb{T})$. Thus $S_+ f \in \text{Dom}(T_\omega)$, and we have

$$S_- T_\Omega S_+ f = S_- \mathbb{P}(zh + c) = S_- (\mathbb{P}(zh) + c) = S_- \mathbb{P}zh = \mathbb{P}z^{-1} \mathbb{P}zh$$

$$= \mathbb{P}z^{-1}\mathbb{P}zh + \mathbb{P}z^{-1}(I_{L_m^p} - \mathbb{P})zh = \mathbb{P}z^{-1}zh = \mathbb{P}h = T_\Omega f,$$

where we used that $\mathbb{P}z^{-1}(I_{L_m^p} - \mathbb{P})g = 0$ for all $g \in L_m^p$. \square

In order to determine the Fredholm properties of T_Ω , via the factorization of Theorem 1.1, we can reduce to the case of a diagonal matrix function in $\text{Rat}^{m \times m}(\mathbb{T})$, with zeroes all on \mathbb{T} . Therefore we will not attempt here to give an explicit description of the kernel, range and domain for the case $\Omega \in \text{Rat}^{m \times m}(\mathbb{T})$ in the form of an analogue of Theorem 2.2 in [12]. For the diagonal matrix case, results are easily obtained by reduction to the scalar case. Here, and in the sequel, we shall identify the direct sum $H^p \oplus \cdots \oplus H^p$ of m copies of H^p with H_m^p , and likewise for L^p .

Proposition 2.4 *Suppose that $\Omega \in \text{Rat}^{m \times m}$ is of the form*

$$\Omega(z) = \text{Diag}(\omega_1(z), \dots, \omega_m(z)), \quad \text{with } \omega_j \in \text{Rat}, j = 1, 2, \dots, m$$

Then

$$\text{Dom}(T_\Omega) = \text{Dom}(T_{\omega_1}) \oplus \text{Dom}(T_{\omega_2}) \oplus \cdots \oplus \text{Dom}(T_{\omega_m}), \quad (2.2)$$

and for $f = f_1 \oplus \cdots \oplus f_m \in \text{Dom}(T_\Omega)$ we have $T_\Omega f = T_{\omega_1} f_1 \oplus \cdots \oplus T_{\omega_m} f_m$. Furthermore, we have

$$\begin{aligned} \text{Ran}(T_\Omega) &= \text{Ran}(T_{\omega_1}) \oplus \text{Ran}(T_{\omega_2}) \oplus \cdots \oplus \text{Ran}(T_{\omega_m}); \\ \text{Ker}(T_\Omega) &= \text{Ker}(T_{\omega_1}) \oplus \text{Ker}(T_{\omega_2}) \oplus \cdots \oplus \text{Ker}(T_{\omega_m}). \end{aligned}$$

Proof For $j = 1, \dots, m$, suppose that $f_j \in \text{Dom}(T_{\omega_j})$. Then $\omega_j f_j = h_j + \rho_j$ with $h_j \in L^p$ and $\rho_j \in \text{Rat}_0(\mathbb{T})$ so that

$$\Omega f = h + \rho$$

where

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in H_m^p, \quad h = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} \in L_m^p, \quad \rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} \in \text{Rat}_0^m(\mathbb{T}). \quad (2.3)$$

Thus $f \in \text{Dom}(T_\Omega)$ and we have $T_\Omega f = T_{\omega_1} f_1 \oplus \cdots \oplus T_{\omega_m} f_m$. It follows that

$$\text{Dom}(T_{\omega_1}) \oplus \text{Dom}(T_{\omega_2}) \oplus \cdots \oplus \text{Dom}(T_{\omega_m}) \subset \text{Dom}(T_\Omega).$$

To show the converse inclusion, suppose that $f \in \text{Dom}(T_\Omega)$. Then there are $h \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$ with $\Omega f = h + \rho$. Decomposing f , h and ρ as in (2.3), it follows that $\omega_j f_j = h_j + \rho_j$ for each j , showing that $f_j \in \text{Dom}(T_{\omega_j})$. Hence (2.2) holds and it follows that the action of T_Ω relates to the action of the operators T_{ω_j} , $j = 1, \dots, m$, as claimed. \square

The formulas for the range and kernel of T_Ω are now straightforward.

Getting an explicit formulation of the domain, range and kernel of T_Ω beyond the diagonal case is much more complicated than in the scalar case, even when $\Omega \in \text{Rat}^{m \times m}(\mathbb{T})$. We indicate the difficulty in the following lemma.

Lemma 2.5 *Let $\Omega \in \text{Rat}^{m \times m}$ and write $\Omega = \Omega_1 + \Omega_2$ where $\Omega_1 \in \text{Rat}^{m \times m}$ with no poles on \mathbb{T} and $\Omega_2 \in \text{Rat}_0^{m \times m}(\mathbb{T})$, so that $\text{Dom}(T_\Omega) = \text{Dom}(T_{\Omega_2})$. Let $\Omega_2 = q^{-1}P$ with $q \in \mathcal{P}$ with zeroes only on \mathbb{T} and $P \in \mathcal{P}^{m \times m}$ so that no root of q is also a root of each entry of P . Suppose $\Omega_2 = \left(\frac{s_{ij}}{q_{ij}}\right)_{i,j=1}^m$ with $s_{ij}, q_{ij} \in \mathcal{P}$ co-prime for all i and j and let q_j be the least common multiple of q_{1j}, \dots, q_{mj} . Then*

$$qH_m^p + \mathcal{P}_{\deg q-1}^m \subset \oplus_{j=1}^m (q_j H^p + \mathcal{P}_{\deg q_j-1}) \subset \text{Dom}(T_\omega) \quad (2.4)$$

and both inclusions can be strict.

Proof The first inclusion follows since the roots of each q_j will be included in the zeroes of q , multiplicities taken into account, which gives $qH^p \subset q_j H^p$. Since $\mathcal{P} \subset qH^p + \mathcal{P}_{\deg q-1}$ and $\mathcal{P} \subset q_j H^p + \mathcal{P}_{\deg q_j-1}$, we obtain

$$qH^p + \mathcal{P}_{\deg q-1} \subset q_j H^p + \mathcal{P}_{\deg q_j-1}, \text{ for } j = 1, \dots, m.$$

For the second inclusion, let $f = f_1 \oplus \dots \oplus f_m \in H_m^p$ and suppose $f_j = q_j h_j + r_j \in q_j H^p + \mathcal{P}_{\deg q_j-1}$. Then $q_j = u_{ij} q_{ij}$ for some polynomial u_{ij} . Write $s_{ij} r_j = q_{ij} r_{ij} + \tilde{r}_j$ for some $\tilde{r}_j \in \mathcal{P}_{\deg q_{ij}-1}$ and $r_{ij} \in \mathcal{P}$. Then

$$s_{ij} f_j = s_{ij} q_j h_j + s_{ij} r_j = q_{ij} (s_{ij} u_{ij} h_j + r_{ij}) + \tilde{r}_j.$$

Since the i th entry in $\Omega_2 f$ is given by $\sum_{j=1}^m \frac{s_{ij}}{q_{ij}} f_j$, we have

$$(\Omega_2 f)_i = \sum_{j=1}^m \frac{s_{ij}}{q_{ij}} f_j = \sum_{j=1}^m (s_{ij} u_{ij} h_j + r_{ij}) + \sum_{j=1}^m \frac{\tilde{r}_j}{q_{ij}}.$$

Note that $\tilde{r}_j/q_{ij} \in \text{Rat}_0(\mathbb{T})$ for each j . Then also $\sum_{j=1}^m \tilde{r}_j/q_{ij} \in \text{Rat}_0(\mathbb{T})$ for each j . This proves that $\Omega f \in H_m^p + \text{Rat}_0^m(\mathbb{T})$, and thus $f \in \text{Dom}(T_\Omega)$.

It is not difficult to construct examples where the first inclusion is strict, use for instance Lemma 3.5 in [11]. To see that the second inclusion can be strict, consider Example 2.6 below. \square

Example 2.6 Consider $\Omega \in \text{Rat}^{2 \times 2}(\mathbb{T})$ given by

$$\Omega(z) = \begin{pmatrix} \frac{z}{z-1} & \frac{1}{z-1} \\ \frac{1}{z+1} & \frac{z+2}{z+1} \end{pmatrix}.$$

Take $f = f_1 \oplus -f_1$ with $f_1 \in H^p$ arbitrarily. Then $\Omega(z)f(z) = f_1 \oplus -f_1 \in H_2^p$, hence $f \in \text{Dom}(T_\Omega)$. In this case, the greatest common divisor of the columns of

Ω is $q_1(z) = z^2 - 1$. By Lemma 3.5 in [11], there exist $f_1 \in H^p$ which are not in $(z - 1)H^p + \mathbb{C}$ (or not in $(z + 1)H^p + \mathbb{C}$) and so $f_1 \notin (z^2 - 1)H^p + \mathcal{P}_1$. Selecting f_1 in such a way, it follows that f is not in $((z^2 - 1)H^p + \mathcal{P}_1) \oplus ((z^2 - 1)H^p + \mathcal{P}_1)$, proving the second inclusion in (2.4) can be strict.

3 Matrix Polynomial Factorization

In this section we prove a few factorization results for matrix polynomials that will be of use in the sequel. The Smith decomposition for matrix polynomials plays a prominent role in our construction, hence for the readers convenience we will list a variation on it here; see Gantmacher [4] or Gohberg–Lancaster–Rodman [10] for a proof. For simplicity, since we only encounter this case, we only consider the case of square matrix polynomials whose determinant is not uniformly zero.

Theorem 3.1 *Let $R \in \mathcal{P}^{m \times m}$ with $\det R(z) \not\equiv 0$. Then we can write*

$$R(z) = E(z)D(z)F(z) \quad (3.1)$$

where E, F are matrix polynomials with nonzero constant determinants and D is a diagonal matrix polynomial that factors as

$$D(z) = D_-(z)D_0(z)D_+(z)$$

with D_- , D_0 and D_+ also diagonal matrix polynomials with zeroes in \mathbb{D} , on \mathbb{T} and outside \mathbb{D} , respectively. Moreover, the diagonal matrix polynomial $D = \text{Diag}(d_1, \dots, d_m)$ can be chosen in such a way that all diagonal entries are monic and d_{j+1} is a factor of d_j for $j = 1, \dots, m - 1$, and with these additional conditions the diagonal entries d_1, \dots, d_m are uniquely determined by R and are given by

$$d_j(z) = \frac{D_j(z)}{D_{j-1}(z)}, \quad j = 1, \dots, m,$$

where $D_0(z) \equiv 1$ and for $r > 0$, D_r is the greatest common divisor of all minors of R of order r . Furthermore, the diagonal entries of D_- , D_0 and D_+ can be taken monic and ordered with respect to factorization, as was done with D , and then D_- , D_0 and D_+ are also uniquely determined by R .

Note that the assumption $\det R(z) \not\equiv 0$ implies that the minors of a given order r cannot all be 0, so that D_r is well defined and not equal to the zero polynomial.

In part of the proofs we require a slightly more refined version of the Smith form, which we shall call the regional Smith form, and which subsumes both the classical Smith form of Theorem 3.1 and the local Smith form (see Theorem S1.10 in [10]); indeed the global and local versions appear in case $\Lambda = \mathbb{C}$ and $\Lambda = \{z_0\}$ for some $z_0 \in \mathbb{C}$, respectively.

Theorem 3.2 *Let $R \in \mathcal{P}^{m \times m}$ with $\det R(z) \not\equiv 0$ and let $\Lambda \subset \mathbb{C}$. Then we can write*

$$R(z) = E_\Lambda(z) D_\Lambda(z) F_\Lambda(z) \quad (3.2)$$

where E_Λ, F_Λ are matrix polynomials which are invertible for all $z \in \Lambda$ and $D_\Lambda = \text{Diag}(p_1, \dots, p_m)$ a diagonal matrix polynomial such that $p_j, j = 1, \dots, m$, has roots only in Λ . Furthermore, the polynomials $p_j, j = 1, \dots, m$ can be chosen to be monic and in such a way that p_{j+1} is a factor of p_j , for $j = 1, \dots, m-1$, and with this additional conditions the diagonal entries p_1, \dots, p_m are uniquely determined by R and Λ .

We refer to the diagonal matrix polynomial D in Theorem 3.2, made unique through the assumptions that the diagonal entries be monic and the division property for subsequent diagonal entries, as the *Smith form of R with respect to Λ* , and any factorization of the type (3.2) with properties as listed in the theorem as a *Smith decomposition of R with respect to Λ* . In the classical case, with $\Lambda = \mathbb{C}$, we simply speak of the Smith form and Smith decomposition of R , without referring to the region.

Proof of Theorem 3.2 The existence of a Smith decomposition of R with respect to Λ follows directly from the classical Smith decomposition of Theorem 3.1. Write $R(z)$ as in (3.1) with $D(z) = \text{Diag}_{j=1}^m(d_j(z))$. Factor each d_j as $d_j(z) = p_j(z)q_j(z)$ with $p_j, q_j \in \mathcal{P}$, p_j monic and having roots only in Λ and q_j having roots only outside Λ . Now set $D_\Lambda(z) = \text{Diag}_{j=1}^m(p_j(z))$, $E_\Lambda = E$ and $F_\Lambda = \text{Diag}_{j=1}^m(q_j(z))F(z)$, but the roots of d_j outside Λ can be distributed over E and F in any other way. It is clear that E_Λ, D_Λ and F_Λ have the required properties, and that the division property of the diagonal entries of D carries over to D_Λ in case the polynomials of D are ordered as in Theorem 3.1.

It remains to prove the uniqueness claim. As in the case of the proof of the local Smith form in [10, Theorem S1.10], this relies on the Cauchy-Binet formula, cf., Subsection 0.8.7 in [15]. Assume that in addition to the factorization (3.2) with properties as listed in the theorem, including the division property of the diagonal entries p_1, \dots, p_m of D , R also admits a second Smith decomposition $R(z) = \tilde{E}_\Lambda(z) \tilde{D}_\Lambda(z) \tilde{F}_\Lambda(z)$ with respect to Λ , with $\tilde{D}_\Lambda(z) = \text{Diag}(\tilde{p}_1(z), \dots, \tilde{p}_m(z))$ and \tilde{p}_{j+1} a factor of \tilde{p}_j for $j = 1, \dots, m-1$. Define $\Phi(z) := E_\Lambda(z)^{-1} \tilde{E}_\Lambda(z)$ and $\Upsilon(z) := \tilde{F}_\Lambda(z) F_\Lambda(z)^{-1}$. Then $\Phi, \Upsilon \in \text{Rat}^{m \times m}$ have no poles or zeroes in Λ . Hence also all the minors of $\Phi, \Upsilon, \Phi^{-1}, \Upsilon^{-1}$ have no poles in Λ . We have

$$D_\Lambda(z) = \Phi(z) \tilde{D}_\Lambda(z) \Upsilon(z) \quad \text{and} \quad \tilde{D}_\Lambda(z) = \Phi(z)^{-1} D_\Lambda(z) \Upsilon(z)^{-1}. \quad (3.3)$$

Write $\underline{m} = \{1, \dots, m\}$ and for $L, S \subset \underline{m}$ with $\#(L) = \#(S)$ and M an $m \times m$ matrix, write $|M|_{L,S}$ for the minor obtained by selecting only the rows indexed by the entries of L and only the columns indexed by the entries of S . Fix $k \in \underline{m}$ and set $L = \{m-k, \dots, m\}$. By the Cauchy-Binet formula

$$\begin{aligned} p_{m-k}(z) \cdots p_m(z) &= |D_\Lambda(z)|_{L,L} = |\Phi(z) \tilde{D}_\Lambda(z) \Upsilon(z)|_{L,L} \\ &= \sum_{S \subset \underline{m}, \#(S)=k} |\Phi(z)|_{L,S} |\tilde{D}_\Lambda(z) \Upsilon(z)|_{S,L} \end{aligned}$$

$$\begin{aligned}
&= \sum_{S \subset \underline{m}, \#(S)=k} |\Phi(z)|_{L,S} |\tilde{D}_\Lambda(z)|_{S,S} |\Upsilon(z)|_{S,L} \\
&= \sum_{S \subset \underline{m}, \#(S)=k} |\Phi(z)|_{L,S} |\Upsilon(z)|_{S,L} \prod_{j \in S} \tilde{p}_j(z).
\end{aligned}$$

Since $|\Phi(z)|_{L,S}$ and $|\Upsilon(z)|_{S,L}$ do not have poles in Λ , $\prod_{j \in S} \tilde{p}_j(z)$ is a factor of the numerator of $|\Phi(z)|_{L,S} |\Upsilon(z)|_{S,L} \prod_{j \in S} \tilde{p}_j(z)$ for all $S \subset \underline{m}$ with $\#(S) = k$. Also, by the factorization order of the diagonal entries in \tilde{D}_Λ we know that $\tilde{p}_{m-k}(z) \cdots \tilde{p}_m(z)$ is a factor of $\prod_{j \in S} \tilde{p}_j(z)$ for all $S \subset \underline{m}$ with $\#(S) = k$. Consequently, by the above identity we know that $\tilde{p}_{m-k}(z) \cdots \tilde{p}_m(z)$ is a factor of $p_{m-k}(z) \cdots p_m(z)$. Applying the same argument to the second identity in (3.3), one obtains that also $p_{m-k}(z) \cdots p_m(z)$ is a factor of $\tilde{p}_{m-k}(z) \cdots \tilde{p}_m(z)$, hence they are equal because both are monic polynomials. Since this identity holds for all $k \in \underline{m}$, it follows that $p_j = \tilde{p}_j$ for $j = 1, \dots, m$. \square

Corollary 3.3 *Let $R \in \mathcal{P}^{m \times m}$ with $\det R(z) \not\equiv 0$ and let $\Lambda \subset \mathbb{C}$. For all matrix polynomials $M, N \in \mathcal{P}^{m \times m}$ which are invertible for all $z \in \Lambda$, the matrix polynomials R and MRN have the same Smith form with respect to Λ .*

Proof Let $R(z) = E(z)D(z)F(z)$ be the factorization of Theorem 3.2 with D the Smith form of R with respect to Λ . Then

$$M(z)R(z)N(z) = (M(z)E(z))D(z)(F(z)N(z))$$

is a factorization of the type in Theorem 3.2, with $M(z)E(z)$ and $F(z)N(z)$ both being invertible for all $z \in \Lambda$ and the diagonal polynomials of D still have the required properties to guarantee uniqueness. In particular D is also the Smith form of MRN with respect to Λ . \square

The next result is used to repair an oversight in the construction in [2].

Lemma 3.4 *Let $F \in \mathcal{P}^{m \times m}$ with $\det F(z) = z^n p(z)$ for a $p \in \mathcal{P}$ with $p(0) \neq 0$. Then*

$$F(z) = Q(z)R(z) \tag{3.4}$$

where $Q, R \in \mathcal{P}^{m \times m}$ with $\det R(z) = p(z)$ and Q a lower triangular matrix polynomial with $\det Q(z) = z^n$. In particular, the diagonal entries of Q are of the form z^{n_1}, \dots, z^{n_m} , with z^{n_j} on the j th diagonal entry, $\sum_{j=1}^m n_j = n$, and, moreover, the indices n_1, \dots, n_m are uniquely determined by F . Furthermore, a factorization (3.4) exists with the polynomial entries left of the diagonal of Q having a degree lower than the degree of the diagonal entry in the same row, and with this additional condition Q and R are uniquely determined.

Proof The proof follows by a recursive procedure, in four parts.

Part 1: First Step Write

$$F(z) = \text{Diag}(z^{n_1}, \dots, z^{n_m}) R_1(z)$$

where $R_1 \in \mathcal{P}^{m \times m}$ and n_i is the highest power of z dividing all the entries in row i . Then $\det R_1(z) = z^{n'} p(z)$ with $n' = n - \sum_{i=1}^m n_i$ and $Q_1(z) := \text{Diag}(z^{n_1}, \dots, z^{n_m})$ is a lower triangular matrix polynomial with $\det Q_1(z) = z^{n-n'}$ with entries left of the diagonal equal to 0, thus of degree 0, hence they have a lower degree than the diagonal entry in the same row. If $n' = 0$ we take $Q = Q_1$ and $R = R_1$ and are done.

Part 2: Second Step In case $n' \neq 0$, write

$$R_1(z) = \begin{pmatrix} r_1(z) \\ \vdots \\ r_m(z) \end{pmatrix}, \text{ with } r_j \in \mathcal{P}^{1 \times m}, j = 1, \dots, m.$$

Then $r_1(0), \dots, r_m(0)$ are linearly dependent. Let k be the smallest integer such that $r_1(0), \dots, r_k(0)$ are linearly dependent. Then there are numbers $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ such that

$$\alpha_1 r_1(0) + \dots + \alpha_{k-1} r_{k-1}(0) + r_k(0) = 0.$$

Put

$$L = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \alpha_1 & \dots & \dots & \alpha_{k-1} & 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

with the α_j 's appearing in the k th row. Then

$$F(z) = \text{Diag}(z^{n_1}, \dots, z^{n_m}) L^{-1} L R_1(z).$$

Since $\sum_{i=1}^k \alpha_i r_i(z)$ has a root at zero, where we set $\alpha_k = 1$, we have

$$L R_1(z) = \begin{pmatrix} r_1(z) \\ \vdots \\ r_{k-1}(z) \\ \sum_{i=1}^k \alpha_i r_i(z) \\ r_{k+1}(z) \\ \vdots \\ r_m(z) \end{pmatrix} = \text{Diag}(1, \dots, 1, z^\ell, 1, \dots, 1) R_2(z)$$

for some $\ell \geq 1$ and $R_2 \in \mathcal{P}^{m \times m}$, where on the right-hand side z^ℓ appears in the k th diagonal entry. Note that

$$L^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ -\alpha_1 & \cdots & \cdots & -\alpha_{k-1} & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad (3.5)$$

again with the α_j 's appearing in the k th row. By direct computation we find

$$\text{Diag}(z^{n_1}, \dots, z^{n_m}) L^{-1} = G_2(z) \text{Diag}(z^{n_1}, \dots, z^{n_m})$$

where

$$G_2(z) := \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ -\alpha_1 z^{n_k - n_1} & \cdots & \cdots & -\alpha_{k-1} z^{n_k - n_{k-1}} & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

with the $-\alpha_j z^{n_k - n_j}$ entries in the k th row. We now have

$$F(z) = G_2(z) \text{Diag}(z^{n_1}, \dots, z^{n_{k-1}}, z^{n_k + \ell}, z^{n_{k+1}}, \dots, z^{n_m}) R_2(z),$$

and $Q_2(z) := G_2(z) \text{Diag}(z^{\tilde{n}_1}, \dots, z^{\tilde{n}_m}) \in \mathcal{P}^{m \times m}$, where we set $\tilde{n}_j = n_j$ if $j \neq k$ and $\tilde{n}_k = n_k + \ell$. Note that $\det R_2(z) = z^{n' - \ell} p(z)$, while the entries in the k th row of Q_2 left of the diagonal have degree n_k and the diagonal entry in the k th row has degree $n_k + \ell > n_k$ and all other off-diagonal entries are 0. In case $n' - \ell = 0$ (equivalently, $\sum_{j=1}^m \tilde{n}_j = n$), take $Q = Q_2$ and $R = R_2$ and we are done.

Part 3: Recursion In case $n' - \ell \neq 0$, repeat the above construction starting with R_2 instead of R_1 . In each step the power of z in the determinant of R_j decreases, hence the process stops after at most n steps. It remains to see that the entries left of the diagonal have the required restriction on the degree. To see that this is the case, we claim that in the j th step, going from factorization $F(z) = Q_{j-1}(z) R_{j-1}(z)$

to $F(z) = Q_j(z)R_j(z)$, all entries of Q_j left of the diagonal have a degree lower than the degree of the diagonal entry in the same row and, in addition, if k is the first integer so that rows 1 to k in $R_{j-1}(0)$ are linearly dependent, then in Q_j all off-diagonal entries in rows $k+1$ to m are 0. These properties certainly hold in the first two steps. Now assume this is satisfied in the step leading to the factorization $F(z) = Q_{j-1}(z)R_{j-1}(z)$. Assume k is the first integer so that rows 1 to k in $R_{j-1}(0)$ are linearly dependent. From the procedure it follows that in the previous step, the first occurrence of linear dependence in the rows of $R_{j-2}(0)$ must also have been in rows 1 to k . Hence, by assumption, in R_{j-1} the off-diagonal entries in rows $k+1$ to m are all 0. Then Q_j is obtained by multiplying Q_{j-1} with a matrix L^{-1} of the form (3.5) on the right and then with $\text{Diag}(1, \dots, 1, z^\ell, 1, \dots, 1)$ also on the right, where z^ℓ appears in the k th entry. One easily checks that rows 1 to $k-1$ of Q_{j-1} and Q_j coincide, due to the lower triangular structure, and that rows $k+1$ to m of Q_{j-1} and Q_j coincide, due to the zeros in the off-diagonal entries in the rows $k+1$ to m . In particular, it follows from the above arguments for all but the k th row that the entries left of the diagonal have a degree less than the diagonal entry in the same row, while all off-diagonal entries in rows $k+1$ to m remain 0. Assume the entries of Q_{j-1} left of the diagonal are given by $q_{i,j}$, $i > j$ and that the i th diagonal entry is z^{n_i} , so that $\deg q_{i,j} < n_i$. Then, on the k th row, left of the diagonal we obtain entries of the form $q_{k,j}(z) - \alpha_j z^{n_j}$ with degree at most n_j , while the diagonal entry becomes $z^{n_j+\ell}$, which proves our claim.

Part 4: Uniqueness We first show that the diagonal entries of Q are unique, without assuming additional degree constraints on the entries left of the diagonal. Suppose that there is another factorization of F of the same type, i.e.,

$$Q(z)R(z) = F(z) = Q'(z)R'(z),$$

with $Q', R' \in \mathcal{P}^{m \times m}$, $\det R'(z) = p(z)$ and Q' lower triangular so that $\det Q'(z) = z^n$. Assume the j th diagonal entries of Q and Q' are z^{n_j} and z^{s_j} , respectively, for $j = 1, \dots, m$. Note that $Q^{-1}(z)$ is in $\text{Rat}^{m \times m}$, lower triangular with z^{-n_j} on the j th diagonal entry, so that

$$R(z)(R')^{-1}(z) = Q^{-1}(z)Q'(z) = \begin{pmatrix} z^{s_1-n_1} & 0 & \dots & 0 \\ * & z^{s_2-n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & z^{s_m-n_m} \end{pmatrix}. \quad (3.6)$$

We have $\sum_{j=1}^m s_j = \sum_{j=1}^m n_j = n$, so we are done if we can prove that $s_j \geq n_j$ for all j . To see that this is the case, multiply (3.6) with p . Since $\det R'(z) = p(z)$, we have $p(z)(R')^{-1}(z) \in \mathcal{P}^{m \times m}$. Hence, the left hand side in (3.6), multiplied with p , is in $\mathcal{P}^{m \times m}$. Consequently, also the right hand side in (3.6), multiplied with p , is in $\mathcal{P}^{m \times m}$. Note that the diagonal entries are of the form $p(z)z^{s_j-n_j}$ and must be in \mathcal{P} . Since $p(0) \neq 0$, this implies $s_j \geq n_j$ for all j , as claimed. Thus $n_j = s_j$ for all j . It follows that the diagonal entries of Q are uniquely determined.

Write $q_{i,j}$ and $q'_{i,j}$ for the (i, j) th entries of Q and Q' , respectively. Now assume $\deg q_{i,j} < n_j$ and $\deg q'_{i,j} < s_j = n_j$. Set $\tilde{R}(z) := R(z)(R')^{-1}(z) \in \text{Rat}^{m \times m}$. We observed above that \tilde{R} is lower triangular with diagonal entries equal to 1, since $s_j = n_j$ for all j , and $p(z)\tilde{R} \in \mathcal{P}^{m \times m}$, so that the entries of \tilde{R} left of the diagonal have the form $\tilde{r}_{i,j}(z)/p(z)$ for a $\tilde{r}_{i,j} \in \mathcal{P}$ for the (i, j) th entry, with $i > j$. Also, we have $Q(z)\tilde{R}(z) = Q'(z)$. We claim that $\tilde{R}(z) = I_m$ for all z , which proves our claim. To see this, we need to show $\tilde{r}_{i,j} = 0$ for all $i > j$. Fix $j \in \{1, \dots, m\}$. Then for $i > j$ the identity $Q(z)\tilde{R}(z) = Q'(z)$ yields

$$q'_{i,j}(z) = q_{i,j}(z) + \sum_{k=j+1}^{i-1} \frac{q_{i,k}(z)\tilde{r}_{k,j}(z)}{p(z)} + \frac{\tilde{r}_{i,j}(z)z^{n_i}}{p(z)}.$$

Recall that $p(0) \neq 0$, so $p(z)$ and z^{n_i} have no common factor. First take $i = j + 1$. In that case we find that $q'_{j+1,j}(z) - q_{j+1,j}(z) = \tilde{r}_{j+1,j}(z)z^{n_{j+1}}/p(z)$. Assume $\tilde{r}_{j+1,j} \neq 0$. Since p and $z^{n_{j+1}}$ have no common factor, the right hand side is a polynomial of degree at least n_{j+1} . However, by assumption the degree of the polynomial on the left hand side is less than n_{j+1} , leading to a contradiction. Hence $\tilde{r}_{j+1,j} = 0$. Now consider $i = j + 2$. Since $\tilde{r}_{j+1,j} = 0$, the above identity for $q'_{j+2,j}$ reduces to $q'_{j+2,j}(z) - q_{j+2,j}(z) = \tilde{r}_{j+2,j}(z)z^{n_{j+2}}/p(z)$, and a similar argument shows that $\tilde{r}_{j+2,j} = 0$. Proceeding this way, one obtains $\tilde{r}_{i,j} = 0$ for all $i > j$, which completes the proof. \square

Lemma 3.5 *Let $P \in \mathcal{P}^{m \times m}$ and $N = \deg P$, so that $\tilde{P}(z) := z^N P\left(\frac{1}{z}\right)$ is in $\mathcal{P}^{m \times m}$. Furthermore, let $P(z) = E(z)D(z)F(z)$ and $\tilde{P}(z) = \tilde{E}(z)\tilde{D}(z)\tilde{F}(z)$ be the Smith decompositions of P and \tilde{P} , so that the diagonal elements p_1, \dots, p_m of P and $\tilde{p}_1, \dots, \tilde{p}_m$ of \tilde{P} are ordered as in Theorem 3.1. Let $\alpha \neq 0$. Then d_j has a root at α of order k if and only if \tilde{d}_j has a root of order k at α^{-1} .*

Proof Let P, \tilde{P} and their Smith decompositions be as stated in the lemma. Recall from Theorem 3.1 that for $j = 1, \dots, m$ we have

$$d_j(z) = \frac{D_j(z)}{D_{j-1}(z)} \quad \text{and} \quad \tilde{d}_j(z) = \frac{\tilde{D}_j(z)}{\tilde{D}_{j-1}(z)}$$

with D_r and \tilde{D}_r the g.c.d. of all minors of order r of P and \tilde{P} , respectively. The relation $\tilde{P}(z) = z^N P\left(\frac{1}{z}\right)$ translates in terms of the g.c.d. of the minors as $\tilde{D}_j(z) = z^{Nj} D_j\left(\frac{1}{z}\right)$, so that

$$\tilde{d}_j(z) = \frac{\tilde{D}_j(z)}{\tilde{D}_{j-1}(z)} = \frac{D_j\left(\frac{1}{z}\right)z^{Nj}}{D_{j-1}\left(\frac{1}{z}\right)z^{N(j-1)}} = z^N d_j\left(\frac{1}{z}\right).$$

From this it directly follows for $\alpha \neq 0$ that d_j has a root of order k at α if and only if \tilde{d}_j has a root of order k at α^{-1} . \square

4 The Wiener–Hopf Type Factorization

In this section we prove the Wiener–Hopf type factorization presented in Theorem 1.1 as well as the uniqueness claims of Theorem 1.2. In fact, we give a construction for how such a factorization can be obtained. The construction relies strongly on the ideas of the construction in Chapter 2 of [2] for contours more general than the unit circle, but without the possibility of having poles on the contour.

Using the polynomial factorization results of Sect. 3, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega(z) \not\equiv 0$. The proof is an adaptation of the proof of Theorem 2.1 in [2] and will be divided into four steps.

Step 1 Firstly, let q be the least common multiple of the denominators of the matrix entries of Ω , so that $q(z)\Omega(z) \in \mathcal{P}^{m \times m}$. As in Lemma 5.1 in [11], write $q^{-1}(z) = z^\kappa \omega_-(z)\omega_o(z)\omega_+(z)$ where $\omega_-(z)$ and $\omega_-(z)^{-1}$ are minus functions, $\omega_+(z)$ and $\omega_+(z)^{-1}$ are plus functions and $\omega_o(z)$ has zeroes and poles only on \mathbb{T} . Note that κ is uniquely determined by q , while the factors ω_- , ω_o , ω_+ are uniquely determined up to a nonzero constant. In fact, if $q(z) = q_-(z)q_o(z)q_+(z)$ with q_- , q_o , $q_+ \in \mathcal{P}$ the factors of q with roots only inside \mathbb{D} , on \mathbb{T} and outside \mathbb{D} , respectively, then $\kappa = -\deg q_-$ and, up to a nonzero constant, $\omega_-(z) = z^{-\kappa}/q_-(z)$, $\omega_o(z) = 1/q_o(z)$ and $\omega_+(z) = 1/q_+(z)$.

Step 2 Define $P_1(z) := q(z)\Omega(z) \in \mathcal{P}^{m \times m}$ and factor P_1 as in the (extended) Smith decomposition of Theorem 3.1:

$$P_1(z) = E_1(z)D_1^-(z)D_1^\circ(z)D_1^+(z)F_1(z),$$

where E_1 and F_1 are matrix polynomials with nonzero constant determinants, and D_1^- , D_1° and D_1^+ are diagonal matrix polynomials, with roots only inside \mathbb{D} , on \mathbb{T} and outside \mathbb{D} , respectively, with in all three the diagonal entries monic and ordered as in Theorem 3.1. Note that $D_1^+(z)F_1(z)$ is a matrix polynomial with all its roots outside \mathbb{D} , hence it is a plus function whose inverse is also a plus function.

Step 3 Note that $E_1(z)D_1^-(z)D_1^\circ(z)$ is a polynomial that has all its zeroes in $\overline{\mathbb{D}}$. Let $N = \deg E_1(z)D_1^-(z)D_1^\circ(z)$ and define

$$P_2(z) := z^N E_1 \left(\frac{1}{z} \right) D_1^- \left(\frac{1}{z} \right) D_1^\circ \left(\frac{1}{z} \right) \in \mathcal{P}^{m \times m},$$

so that

$$P_1(z) = z^N P_2 \left(\frac{1}{z} \right) D_1^+(z)F_1(z).$$

Then the zeroes of P_2 can only be on \mathbb{T} , outside $\overline{\mathbb{D}}$ (except at ∞) or at 0. Next determine the Smith decomposition of Theorem 3.1 for P_2 :

$$P_2(z) = E_2(z) D_2^-(z) D_2^\circ(z) D_2^+(z) F_2(z),$$

where E_2 and F_2 are matrix polynomials with nonzero constant determinants, and D_2^- , D_2° and D_2^+ are diagonal matrix polynomials, with roots only inside \mathbb{D} , on \mathbb{T} and outside $\overline{\mathbb{D}}$, respectively, with in all three the diagonal entries monic and ordered as in Theorem 3.1.

Since 0 is the only root of P_2 in \mathbb{D} , there exist $\rho_1 \geq \rho_2 \geq \dots \geq \rho_m \geq 0$ so that $D_2^-(z) = \text{Diag}(z^{\rho_1}, \dots, z^{\rho_m})$. Moreover, since D_2° is a diagonal matrix polynomial whose diagonal entries are monic polynomials with roots only on \mathbb{T} , we can write $D_2^\circ\left(\frac{1}{z}\right) = \tilde{D}_2^\circ(z) \tilde{D}_2^-\left(\frac{1}{z}\right)$ with $\tilde{D}_2^\circ, \tilde{D}_2^- \in \mathcal{P}^{m \times m}$ diagonal matrix polynomials with \tilde{D}_2° having monic diagonal entries with roots only on \mathbb{T} and \tilde{D}_2^- having roots only at zero. In fact, if p_j is the j th diagonal entry of D_2° , then the j th diagonal entry of \tilde{D}_2° is uniquely determined and given by $\frac{z^{\deg p_j}}{p_j(0)} p_j\left(\frac{1}{z}\right)$, while the j th diagonal entry of $\tilde{D}_2^-\left(\frac{1}{z}\right)$ is equal to $\frac{p_j(0)}{z^{\deg p_j}}$. In particular, $\tilde{D}_2^-(z) = \text{Diag}(p_1(0)^{-1} z^{\eta_1}, \dots, p_m(0)^{-1} z^{\eta_m})$ for integers $\eta_1 \geq \eta_2 \geq \dots \geq \eta_m \geq 0$, since by construction $\deg p_j \geq \deg p_{j+1}$ for $j = 1, \dots, m-1$. We now obtain that

$$\begin{aligned} P_1(z) &= z^N E_2\left(\frac{1}{z}\right) D_2^-\left(\frac{1}{z}\right) D_2^\circ\left(\frac{1}{z}\right) D_2^+\left(\frac{1}{z}\right) F_2\left(\frac{1}{z}\right) D_1^+(z) F_1(z) \\ &= z^N E_2\left(\frac{1}{z}\right) D_2^+\left(\frac{1}{z}\right) D_2^\circ\left(\frac{1}{z}\right) D_2^-\left(\frac{1}{z}\right) F_2\left(\frac{1}{z}\right) D_1^+(z) F_1(z) \\ &= z^N E_2\left(\frac{1}{z}\right) D_2^+\left(\frac{1}{z}\right) \tilde{D}_2^\circ(z) \tilde{D}_2^-\left(\frac{1}{z}\right) D_2^-\left(\frac{1}{z}\right) F_2\left(\frac{1}{z}\right) D_1^+(z) F_1(z). \end{aligned} \quad (4.1)$$

Since E_2 has a constant and nonzero determinant, for $z \neq 0$, $\det E_2\left(\frac{1}{z}\right)$ is also constant and nonzero, so that $E_2\left(\frac{1}{z}\right) \in \text{Rat}^{m \times m}$ can only have zeroes and poles at 0. Hence $E_2\left(\frac{1}{z}\right)$ is a minus function whose inverse is also a minus function. Furthermore, since $D_2^+(z) \in \mathcal{P}^{m \times m}$ has only zeroes outside $\overline{\mathbb{D}}$, the (diagonal) rational matrix function $D_2^+\left(\frac{1}{z}\right)$ has zeroes only inside \mathbb{D} and can only have a pole at 0. Thus $D_2^+\left(\frac{1}{z}\right)$ is a minus function whose inverse is also a minus function. Hence the same conclusion holds for $E_2\left(\frac{1}{z}\right) D_2^+\left(\frac{1}{z}\right)$.

Step 4 Let $K > 0$ be the smallest integer such that

$$P_3(z) := z^K \tilde{D}_2^-\left(\frac{1}{z}\right) D_2^-\left(\frac{1}{z}\right) F_2\left(\frac{1}{z}\right) \in \mathcal{P}^{m \times m}.$$

Note that $\det \tilde{D}_2^-(z) D_2^- \left(\frac{1}{z} \right) F_2 \left(\frac{1}{z} \right) = cz^{-\xi}$ for some constant c and with $\xi = \sum_{k=1}^m (\rho_k + \eta_k)$, so that $\det P_3(z) = cz^n$, with $n := mK - \xi$ being nonnegative by choice of K . Now apply Lemma 3.4 to P_3 . It follows that we can write $P_3(z) = Q_3(z)F_3(z)$ with $Q_3, F_3 \in \mathcal{P}^{m \times m}$ with F_3 having a nonzero constant determinant and Q_3 lower triangular with $\det Q_3(z) = z^n$. Inserting this into (4.1) yields

$$P_1(z) = z^{N-K} E_2 \left(\frac{1}{z} \right) D_2^+ \left(\frac{1}{z} \right) \tilde{D}_2^\circ(z) Q_3(z) F_3(z) D_1^+(z) F_1(z).$$

Using that $\Omega(z) = q(z)^{-1} P_1(z)$ along with the factorization of q^{-1} in Step 1, we obtain that

$$\Omega(z) = z^{-k} \Omega_-(z) \Omega_\circ(z) P_0(z) \Omega_+(z), \quad (4.2)$$

where in case $N - K + \kappa \leq 0$ we set $k = -(N - K + \kappa)$ and define

$$\begin{aligned} \Omega_-(z) &:= \omega_-(z) E_2 \left(\frac{1}{z} \right) D_2^+ \left(\frac{1}{z} \right), \quad \Omega_\circ(z) := \omega_\circ(z) \tilde{D}_2^\circ(z), \\ P_0(z) &:= Q_3(z), \quad \Omega_+(z) := \omega_+(z) F_3(z) D_1^+(z) F_1(z), \end{aligned} \quad (4.3)$$

while if $N - K + \kappa > 0$ we set $k = 0$, define $\Omega_-, \Omega_\circ, \Omega_+$ as above and take $P_0(z) = z^{N-K+\kappa} Q_3(z)$.

Since both $\omega_-(z)$ and $E_2 \left(\frac{1}{z} \right) D_2^+ \left(\frac{1}{z} \right)$ are minus function whose inverses are minus functions, the same is true for $\Omega_-(z)$. That $P_0(z)$ has the required form follows directly from Lemma 3.4. It is also straightforward from the construction that $\Omega_\circ(z)$ is a diagonal matrix whose diagonal entries are scalar rational functions with poles and zeroes only on \mathbb{T} . Finally, all factors of $\Omega_+(z)$ are plus functions whose inverses are also plus functions, hence $\Omega_+(z)$ also has this property. We conclude that the factorization (4.2)–(4.3) of Ω has the required properties. \square

Proof of Theorem 1.2 Upon inspection of the proof of Theorem 1.1, and specifically (4.3) using the definitions of ω_- , ω_\circ and ω_+ from Step 1 of the proof, it follows that P_+ and P_- in (1.3), that is, $P_+(z) = F_3(z) D_1^+(z) F_1(z)$ and $P_-(z) = E_2(z) D_2^+(z)$, are matrix polynomials with the required properties. It also follows that for D_\circ in (1.3), that is, $D_\circ(z) = \tilde{D}_2^\circ(z)$ as defined in Step 3 of the proof, the required ordering of the diagonal entries carries over from the corresponding ordering of D_2° from which \tilde{D}_2° is constructed. It is also clear from the construction, based on Lemma 3.4, that we may take P_0 to have the required form. Finally, assume $k > 0$, that is, $N - K + \kappa < 0$. In case $P_0(0) = 0$, we can write $P_0(z) = z^j \hat{P}_0(z)$ for some $j > 0$ and $\hat{P}_0 \in \mathcal{P}^{m \times m}$ with $\hat{P}_0(0) \neq 0$. In case $k \geq j$, replace k by $k - j$ and P_0 by \hat{P}_0 and in case $k < j$, replace k by 0 and P_0 by $z^{-k} P_0(z) = z^{j-k} \hat{P}_0(z)$. In both cases the adjusted k and P_0 have the required relation, while the structure of P_0 is maintained. It follows that the factorization obtained from the construction in the proof of Theorem 1.1, with the small modifications described here, has the required form.

Let (1.2) be a factorization of Ω satisfying the conditions of Theorem 1.1 as well as the additional conditions of Theorem 1.2. where P_+ , D_\circ and P_- are defined as in

(1.3). Assume that a second factorization of this type exists:

$$\Omega(z) = z^{-\tilde{k}} \tilde{\Omega}_-(z) \tilde{\Omega}_o(z) \tilde{P}_0(z) \tilde{\Omega}_+(z), \quad (4.4)$$

and write \tilde{P}_+ , \tilde{D}_o and \tilde{P}_- for the matrix polynomials constructed via (1.3) for the factorization (4.4). It then follows that

$$z^{-k} P_- \left(\frac{1}{z} \right) D_o(z) P_0(z) P_+(z) = z^{-\tilde{k}} \tilde{P}_- \left(\frac{1}{z} \right) \tilde{D}_o(z) \tilde{P}_0(z) \tilde{P}_+(z).$$

Hence, for each integer $L \geq 0$ we have

$$\begin{aligned} & z^{\tilde{k}+L} \det \left(P_- \left(\frac{1}{z} \right) \right) D_o(z) P_0(z) P_+(z) (\det(\tilde{P}_+(z)) \tilde{P}_+(z)^{-1}) \\ &= z^{k+L} \left(\det \left(P_- \left(\frac{1}{z} \right) \right) P_- \left(\frac{1}{z} \right)^{-1} \right) \tilde{P}_- \left(\frac{1}{z} \right) \tilde{D}_o(z) \tilde{P}_0(z) \det(\tilde{P}_+(z)). \end{aligned}$$

Now choose L large enough so that both $z^{k+L} (\det(P_- (\frac{1}{z})) P_- (\frac{1}{z})^{-1}) \tilde{P}_- (\frac{1}{z})$ and $z^{\tilde{k}+L} \det(P_- (\frac{1}{z}))$ are polynomials, so that on both sides of the identity we have a factorization of polynomials. Now observe that all four polynomials

$$\begin{aligned} & z^{\tilde{k}+L} \det \left(P_- \left(\frac{1}{z} \right) \right), \quad P_0(z) P_+(z) (\det(\tilde{P}_+(z)) \tilde{P}_+(z)^{-1}), \\ & z^{k+L} \left(\det \left(P_- \left(\frac{1}{z} \right) \right) P_- \left(\frac{1}{z} \right)^{-1} \right) \tilde{P}_- \left(\frac{1}{z} \right), \quad \tilde{P}_0(z) \det(\tilde{P}_+(z)), \end{aligned}$$

do not have roots on \mathbb{T} . Since the diagonal entries of D_o and \tilde{D}_o are ordered as stated in Theorem 1.2, it now follows from the uniqueness claim of Theorem 3.2 that both D_o and \tilde{D}_o are equal to the Smith form of the above polynomial with respect to \mathbb{T} . In particular, D_o and \tilde{D}_o coincide. \square

Corollary 4.1 *Let $\Omega \in \text{Rat}^{m \times m}$ factor as (1.2) with the factors as in Theorem 1.1 satisfying the additional conditions of Theorem 1.2. Let q be the least common multiple of the denominators of the matrix entries of Ω . Then D_o defined in (1.3) is equal to the Smith form of $P_1(z) := q(z)\Omega(z) \in \mathcal{P}^{m \times m}$ with respect to \mathbb{T} .*

Proof By the uniqueness claim of Theorem 1.2 and the fact that the construction in the proof of Theorem 1.1 leads to a factorization that satisfies the additional conditions of Theorem 1.2, it suffices to prove that the factor D_o in (1.3) obtained from the construction of Theorem 1.1 coincides with the Smith form of P_1 with respect to \mathbb{T} . In this case, $D_o(z) = \tilde{D}_2^\circ(z)$ as constructed in Step 3 of the proof of Theorem 1.1, while D_1° from Step 2 of the proof is the Smith form of P_1 with respect to \mathbb{T} . Hence we need to show that $D_1^\circ = \tilde{D}_2^\circ$.

We now follow the various steps of the construction of \tilde{D}_2° in the proof of Theorem 1.1. Set $\tilde{P}_2(z) := E_1(z)D_1^-(z)D_1^+(z) \in \mathcal{P}^{m \times m}$. Since $P_1(z) = \tilde{P}_2(z)D_1^+(z)F_1(z)$ and $D_1^+(z)F_1(z) \in \mathcal{P}^{m \times m}$ is invertible for all $z \in \mathbb{T}$, it follows from Corollary 3.3 that D_1^+ is also the Smith form of \tilde{P}_2 with respect to \mathbb{T} . Note that P_2 defined in Step 2 is given by $P_2(z) = z^{\deg \tilde{P}_2} \tilde{P}_2\left(\frac{1}{z}\right)$, and that D_2° in Step 2 is the Smith form of P_0 with respect to \mathbb{T} . The relation between D_1° and D_2° follows from Lemma 3.5. Say $D_1^\circ(z) = \text{Diag}(d_1^\circ(z), \dots, d_m^\circ(z))$ and $D_2^\circ(z) = \text{Diag}(p_1^\circ(z), \dots, p_m^\circ(z))$ with $d_j^\circ, p_j^\circ \in \mathcal{P}$ monic and with roots only on \mathbb{T} , for $j = 1, \dots, m$. Then $d_j^\circ(0) \neq 0$, $\deg d_j^\circ = \deg p_j^\circ$ and $p_j^\circ(z) = \frac{z^{\deg d_j^\circ}}{d_j^\circ(0)} d_j^\circ\left(\frac{1}{z}\right)$. On the other hand, the j th diagonal entry of \tilde{D}_2° is given by

$$\frac{z^{\deg p_j^\circ}}{p_j^\circ(0)} p_j^\circ\left(\frac{1}{z}\right) = \frac{z^{\deg p_j^\circ}}{p_j^\circ(0)} \frac{z^{-\deg d_j^\circ}}{d_j^\circ(0)} d_j^\circ(z) = \frac{1}{p_j^\circ(0)d_j^\circ(0)} d_j^\circ(z) = d_j^\circ(z),$$

using $\deg p_j^\circ = \deg d_j^\circ$ in the first identity and the fact that both d_j° and the resulting polynomial are monic (so that $p_j^\circ(0)d_j^\circ(0)$ must be 1) in the second identity. Consequently, d_j° is the j th diagonal entry of \tilde{D}_2° , and thus $\tilde{D}_2^\circ = D_1^\circ$, as claimed. \square

Corollary 4.2 *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega(z) \neq 0$ and suppose*

$$\Omega(z) = z^{-k} \Omega_-(z) \Omega_o(z) P_0(z) \Omega_+(z)$$

is the factorization of Ω as in Theorem 1.1. Then the zeroes and poles of Ω on \mathbb{T} correspond to the zeroes and poles of ψ_j where $\Omega_o(z) = \text{Diag}(\psi_j(z))_{j=1}^m$.

5 An Example

In this section we present an example illustrating the factorization procedure. Consider

$$\Omega(z) = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix}.$$

Then $q(z) = z - 1$ and we have $q(z)^{-1} = (z - 1)^{-1} = z^0 \omega_-(z) \omega_o(z) \omega_+(z)$ with $\omega_-(z) = \omega_+(z) = 1$ and $\omega_o(z) = (z - 1)^{-1}$. The Smith decomposition of $P_1(z) = q(z)\Omega(z) = \begin{pmatrix} z^{-1} & 1 \\ 0 & z-1 \end{pmatrix}$ is given by

$$\begin{aligned} P_1(z) &= E_1(z)D_1(z)F_1(z) = E_1(z)D_1^-(z)D_1^\circ(z)D_1^+(z)F_1(z) \\ &= \begin{pmatrix} 0 & 1 \\ -1 & z-1 \end{pmatrix} \begin{pmatrix} (z-1)^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z-1 & 1 \end{pmatrix} \end{aligned}$$

with $D_1^-(z) = D_1^+(z) = I_2$. Then $E_1(z)D_1^-(z)D_1^\circ(z) = \begin{pmatrix} 0 & 1 \\ -(z-1)^2 & z-1 \end{pmatrix}$, and so

$$\begin{aligned} P_2(z) &= z^N E_1 \left(\frac{1}{z} \right) D_1^- \left(\frac{1}{z} \right) D_1^\circ \left(\frac{1}{z} \right) \\ &= z^2 \begin{pmatrix} 0 & 1 \\ -(\frac{1}{z}-1)^2 & \frac{1}{z}-1 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ -(z-1)^2 & z-z^2 \end{pmatrix}. \end{aligned}$$

The Smith decomposition of P_2 is given by

$$\begin{aligned} P_2(z) &= E_2(z)D_2(z)F_2(z) \\ &= \begin{pmatrix} -z-1 & -z^2 \\ 1 & z-1 \end{pmatrix} \begin{pmatrix} (z-1)^2 z^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ (z-1)^2(z+1) & z(z^2-z-1) \end{pmatrix}. \end{aligned}$$

Hence

$$D_2^-(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2^\circ(z) = \begin{pmatrix} (z-1)^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D_2^+(z) = I_2.$$

Put

$$D_2^\circ \left(\frac{1}{z} \right) = \begin{pmatrix} \left(\frac{1}{z}-1 \right)^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1-z)^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix} = \tilde{D}_2^\circ(z) \tilde{D}_2^- \left(\frac{1}{z} \right).$$

Then

$$\begin{aligned} P_3(z) &= z^K \tilde{D}_2^-(z) D_2^- \left(\frac{1}{z} \right) F_2 \left(\frac{1}{z} \right) \\ &= z^4 \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ \left(\frac{1}{z}-1 \right)^2 \left(\frac{1}{z}+1 \right) & \frac{1}{z} \left(\frac{1}{z^2}-\frac{1}{z}-1 \right) \end{pmatrix}, \end{aligned}$$

from which it follows that

$$P_3(z) = \begin{pmatrix} -1 & -1 \\ z(1-z)^2(1+z) & z(1-z-z^2) \end{pmatrix} \quad \text{and thus} \quad \det P_3(z) = z^4.$$

The above computations conclude Steps 1, 2 and 3 of the procedure in the proof of Theorem 1.1. To conclude Step 4 we have to apply the recursive procedure from the proof of Lemma 3.2 to factor P_3 .

In the first step we factor

$$P_3(z) = Q_1(z)R_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^3 - z^2 - z + 1 & 1 - z - z^2 \end{pmatrix}.$$

The sum of the diagonal multiplicities is 1 which is less than 4. Note that $r_1(0) = (-1 \ -1)$ and $r_2(0) = (1 \ 1)$. Hence we find $L_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{aligned} P_3(z) &= Q_1(z)L_1^{-1}L_1R_1(z) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z^3 - z^2 - z + 1 & -1 \\ 1 - z - z^2 & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z^3 - z^2 - z & -1 \\ -z - z^2 & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} z^2 - z - 1 & -1 \\ -1 - z & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & z^2 \end{pmatrix} \begin{pmatrix} z^2 - z - 1 & -1 \\ -1 - z & \end{pmatrix} = Q_2(z)R_2(z). \end{aligned}$$

The sum of the diagonal multiplicities is 2, less than 4. The rows of $R_2(0)$ are $r'_1(0) = r'_2(0) = (-1 \ -1)$. Hence we take $L_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Then

$$\begin{aligned} P_3(z) &= Q_2(z)R_2(z) = Q_2(z)L_2^{-1}L_2R_2(z) \\ &= \begin{pmatrix} 1 & 0 \\ -z & z^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z^2 - z - 1 & -1 \\ -1 - z & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & z^2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z^2 - z & -z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z & z^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z - 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^2 - z & z^3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z - 1 & -1 \end{pmatrix} = Q_3(z)R_3(z). \end{aligned}$$

The sum of the diagonal multiplicities is 3, still less than 4, and so we apply the procedure one more time. Now $r''_1(0) = r''_2(0)(-1 \ -1)$ and so we have $L_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Then

$$\begin{aligned} P_3(z) &= Q_3(z)R_3(z) = Q_3(z)L_3^{-1}L_3R_3(z) \\ &= \begin{pmatrix} 1 & 0 \\ z^2 - z & z^3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z - 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^3 + z^2 - z & z^3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ z & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^3 + z^2 - z & z^4 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = Q_4(z)R_4(z). \end{aligned}$$

This provides the factorization of P_3 from Lemma 3.4. We have now computed all required matrix functions for the factorization of Ω in (4.2)–(4.3). This yields

$$\Omega(z) = z^{N-K+\kappa} \Omega_-(z) \Omega_o(z) P_0(z) \Omega_+(z),$$

with $N - K + \kappa = 2 - 4 + 0 = -2$ and where

$$\begin{aligned}\Omega_-(z) &= \omega_-(z)E_2\left(\frac{1}{z}\right)D_2^+\left(\frac{1}{z}\right) = E_2\left(\frac{1}{z}\right) = \begin{pmatrix} -\frac{1}{z} - 1 & -\frac{1}{z^2} \\ 1 & \frac{1}{z-1} \end{pmatrix}, \\ \Omega_o(z) &= \omega_o(z)\tilde{D}_2^o(z) = \frac{1}{z-1} \begin{pmatrix} (1-z)^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z-1 & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix}, \\ \Omega_+(z) &= \omega_+(z)R_4(z)D_1^+(z)F_1(z) = R_4(z)F_1(z) = \begin{pmatrix} -z & -1 \\ 1 & 0 \end{pmatrix}, \\ \text{and } P_0(z) &= Q_4(z) = \begin{pmatrix} 1 & 0 \\ z^3 + z^2 - z & z^4 \end{pmatrix}.\end{aligned}$$

6 Factorization of the Toeplitz Operator

In this section we prove Theorem 1.3 by using the Wiener–Hopf type factorization of Theorem 1.1. We first prove some technical lemmas.

Lemma 6.1 *Suppose that $\Omega, U, V \in \text{Rat}^{m \times m}$ with U a minus function whose inverse is a minus function and V a plus function. Then $T_{\Omega V} = T_{\Omega}T_V$ and $T_{U\Omega} = T_UT_{\Omega}$.*

Proof Let $f \in \text{Dom}(T_{\Omega V})$. Then $\Omega Vf = h + \rho$ for $h \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$ and $T_{\Omega V}f = \mathbb{P}h$. Since V has no poles in $\overline{\mathbb{D}}$, V is analytic and bounded on \mathbb{D} . Therefore, T_V is bounded and $T_Vf = Vf$. Put $g = T_Vf = Vf$. Then by the above relation we have $g \in \text{Dom}(T_{\Omega})$ and $T_{\Omega}g = \mathbb{P}h$. In other words, $T_{\Omega}T_Vf = T_{\Omega V}f$. So T_V maps $\text{Dom}(T_{\Omega V})$ into $\text{Dom}(T_{\Omega})$ and on $\text{Dom}(T_{\Omega V})$ the operators $T_{\Omega V}$ and $T_{\Omega}T_V$ coincide. To see that $T_{\Omega V} = T_{\Omega}T_V$, suppose that $u \in \text{Dom}(T_{\Omega}T_V)$. Since V is bounded and analytic on \mathbb{D} , we have $Vu = T_Vu \in \text{Dom}(T_{\Omega})$, that is, $\Omega Vu = w + \eta$ for some $w \in L_m^p$ and $\eta \in \text{Rat}_0^m(\mathbb{T})$, and so $u \in \text{Dom}(T_{\Omega V})$. This proves that $\text{Dom}(T_{\Omega}T_V) = \text{Dom}(T_{\Omega V})$, and hence that $T_{\Omega}T_V = T_{\Omega V}$.

Next we prove that $T_{U\Omega} = T_UT_{\Omega}$. Let $f \in \text{Dom}(T_{U\Omega})$. Then $U\Omega f = h + \rho$ with $h \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$. Hence $\Omega f = U^{-1}h + U^{-1}\rho$. Since U^{-1} is minus function, it is analytic outside $\overline{\mathbb{D}}$ and thus $U^{-1}\rho$ can be written as $h_1 + \rho_1$ with h_1 a rational vector function in L_m^p with poles only in the unit disc and $\rho_1 \in \text{Rat}_0^m(\mathbb{T})$. In particular, $\mathbb{P}h_1 = 0$. So $\Omega f = U^{-1}h + h_1 + \rho_1$, which shows that $f \in \text{Dom}(T_{\Omega})$ and $T_{\Omega}f = \mathbb{P}(U^{-1}h + h_1) = \mathbb{P}(U^{-1}h)$. Since U is a minus function, we have $\mathbb{P}U(I - \mathbb{P})(U^{-1}h) = 0$. Therefore, we find that

$$\begin{aligned}T_UT_{\Omega}f &= T_U\mathbb{P}(U^{-1}h) = \mathbb{P}U\mathbb{P}(U^{-1}h) = \mathbb{P}U\mathbb{P}(U^{-1}h) + \mathbb{P}U(I - \mathbb{P})(U^{-1}h) \\ &= \mathbb{P}UU^{-1}h = \mathbb{P}h = T_{U\Omega}f.\end{aligned}$$

We proved that $\text{Dom}(T_{U\Omega}) \subset \text{Dom}(T_{\Omega}) = \text{Dom}(T_UT_{\Omega})$ and that $T_{U\Omega}$ and T_UT_V coincide on $\text{Dom}(T_{U\Omega})$. It remains to prove $\text{Dom}(T_UT_{\Omega}) \subset \text{Dom}(T_{U\Omega})$. Let $v \in \text{Dom}(T_UT_{\Omega}) = \text{Dom}(T_{\Omega})$. Then $\Omega v = w + \eta$ for $w \in L_m^p$ and $\eta \in \text{Rat}_0^m(\mathbb{T})$. Then $U\Omega v = Uw + U\eta$ and because U is a minus function, $Uw \in L_m^p$ and $U\eta = w' + \eta'$

for $w' \in L_m^p$ and $\eta' \in \text{Rat}_0^m(\mathbb{T})$. Hence $U\Omega v = Uw + w' + \eta' \in L_m^p + \text{Rat}_0^m(\mathbb{T})$, so that $v \in \text{Dom}(T_{U\Omega})$. \square

Lemma 6.2 *Let $\Omega \in \text{Rat}^{m \times m}(\mathbb{T})$. Then for $k \geq 0$, $T_{z^{-k}\Omega} = T_{z^{-k}I_m}T_\Omega$.*

Proof It suffices to show that $\text{Dom}(T_\Omega) = \text{Dom}(T_{z^{-k}\Omega})$ and that $T_{z^{-k}\Omega}$ and $T_{z^{-k}I_m}T_\Omega$ coincide on $\text{Dom}(T_\Omega)$. Suppose that $f \in \text{Dom}(T_{z^{-k}\Omega})$. Then $z^{-k}\Omega f = h + \rho$ with $h \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$. Then $\Omega f = z^k h + z^k \rho$ and $z^k h$ is still in L_m^p . Apply the Euclidian algorithm entrywise to write $z^k \rho = \rho_1 + \rho_2$ with $\rho_2 \in \text{Rat}_0^m(\mathbb{T})$ and $\rho_1 \in \mathcal{P}_{k-1}^m$. Then $\Omega f = (z^k h + \rho_1) + \rho_2 \in L_m^p + \text{Rat}_0^m(\mathbb{T})$ from which it follows that $f \in \text{Dom}(T_\Omega)$. Let $r \in \mathcal{P}_{k-1}^m$ so that $\mathbb{P}z^k h = z^k \mathbb{P}h + r$. Then we have

$$\begin{aligned} T_{z^{-k}I_m}T_\Omega f &= T_{z^{-k}I_m}\mathbb{P}(z^k h + \rho_1) = T_{z^{-k}I_m}(z^k \mathbb{P}h + r + \rho_1) \\ &= T_{z^{-k}I_m}z^k \mathbb{P}h = \mathbb{P}h = T_{z^{-k}\Omega}f. \end{aligned}$$

Thus $\text{Dom}(T_{z^{-k}\Omega}) \subset \text{Dom}(T_\Omega)$ and the operators $T_{z^{-k}I_m}T_\Omega$ and $T_{z^{-k}\Omega}$ coincide on $\text{Dom}(T_{z^{-k}\Omega})$. To prove the converse inclusion, suppose $u \in \text{Dom}(T_\Omega)$. Then $\Omega u = w + \eta$ for $w \in L_m^p$ and $\eta \in \text{Rat}_0^m(\mathbb{T})$, so that

$$z^{-k}\Omega u = z^{-k}w + z^{-k}\eta \in L_m^p + \text{Rat}_0^m(\mathbb{T} \cup \{0\}).$$

Now write $z^{-k}\eta = \eta_1 + \eta_2$ with $\eta_2 \in \text{Rat}_0^m(\mathbb{T})$ and $\eta_1 \in \text{Rat}^m$, with only a pole at 0. Then $\eta_1 \in L_m^p$ and $\mathbb{P}\eta_1 = 0$. We now have $z^{-k}\Omega u = (z^{-k}w + \eta_1) + \eta_2 \in L_m^p + \text{Rat}_0^m(\mathbb{T})$ and so $u \in \text{Dom}(T_{z^{-k}\Omega})$. \square

Proof of Theorem 1.3 Factor $\Omega = z^{-k}\Omega_- \Omega_\circ P_0 \Omega_+$ as in Theorem 1.1. Then $P_0 \Omega_+$ is a plus function, and Ω_- is a minus function whose inverse is also a minus function. Thus, by Lemma 6.1, we can write $T_\Omega = T_{\Omega_-} T_{z^{-k}\Omega_\circ} T_{P_0} T_{\Omega_+}$. Applying Lemma 6.2 we have $T_\Omega = T_{\Omega_-} T_{z^{-k}I_m} T_{\Omega_\circ} T_{P_0} T_{\Omega_+}$. \square

7 Fredholm Properties

Using the Wiener–Hopf type factorization from Theorem 1.1 and the corresponding factorization of Toeplitz operators in Theorem 1.3 we are now in a position to prove Theorem 1.4 via a partial reduction to the diagonal case.

Proof of Theorem 1.4 Let $\Omega \in \text{Rat}^{m \times m}$ be factored as in Theorem 1.1:

$$\Omega(z) = z^{-k}\Omega_-(z)\Omega_\circ(z)P_0(z)\Omega_+(z),$$

with $k \geq 0$, $\Omega_+, \Omega_-, \Omega_\circ \in \text{Rat}^{m \times m}$ so that Ω_- a minus function whose inverse is a minus function, Ω_+ a plus function whose inverse is a plus function, $\Omega_\circ = \text{Diag}(\phi_1, \dots, \phi_m)$ a diagonal matrix function whose entries have poles and zeros only on \mathbb{T} and P_0 a lower triangular polynomial matrix with $\det P_0(z) = z^n$ for some integer $n \geq 0$. Applying Theorem 1.3 gives

$$T_\Omega = T_{\Omega_-} T_{z^{-k}} T_{\Omega_\circ} T_{P_0} T_{\Omega_+}.$$

Given that Ω_- and its inverse are minus functions, T_{Ω_-} is invertible with $T_{\Omega_-}^{-1} = T_{\Omega_-^{-1}}$. Similarly, T_{Ω_+} is invertible and $T_{\Omega_+}^{-1} = T_{\Omega_+^{-1}}$. Thus T_{Ω_-} , $T_{\Omega_-^{-1}}$, T_{Ω_+} and $T_{\Omega_+^{-1}}$ are all Fredholm with index 0 and

$$T_{z^{-k}I_m} T_{\Omega_0} T_{P_0} = T_{\Omega_-^{-1}} T_{\Omega} T_{\Omega_+^{-1}}, \quad T_{\Omega} = T_{\Omega_-} T_{z^{-k}I_m} T_{\Omega_0} T_{P_0} T_{\Omega_+}.$$

Applying item (iii) of Theorem IV.2.7 from [5] (see also [8]) it now follows that T_{Ω} is Fredholm if and only if $T_{z^{-k}I_m} T_{\Omega_0} T_{P_0}$ is Fredholm and in that case we have

$$\text{Index}(T_{\Omega}) = \text{Index}(T_{z^{-k}I_m} T_{\Omega_0} T_{P_0}).$$

Since $P_0(z)$ is a matrix polynomial with zeroes only at 0, T_{P_0} is a bounded Fredholm operator. Therefore, if $T_{z^{-k}I_m} T_{\Omega_0} T_{P_0}$ is Fredholm, then we find that $T_{z^{-k}I_m} T_{\Omega_0}$ is Fredholm, by Theorem 3.4 of [17], while conversely, if $T_{z^{-k}I_m} T_{\Omega_0}$ is Fredholm, then $T_{z^{-k}I_m} T_{\Omega_0} T_{P_0}$ is Fredholm by Theorem IV.2.7 from [5]. Furthermore, in this case we have

$$\text{Index}(T_{z^{-k}I_m} T_{\Omega_0} T_{P_0}) = \text{Index}(T_{z^{-k}I_m} T_{\Omega_0}) + \text{Index}(T_{P_0}).$$

Moreover, consider the Wiener–Hopf factorization $P_0(z) = \Phi_-(z) D_0(z) \Phi_+(z)$ of P_0 , cf., Theorem XXIV.3.1 in [7], with Φ_- and Φ_-^{-1} minus functions, Φ_+ and Φ_+^{-1} plus functions, and $D_0(z) = \text{Diag}(z^{k_1}, \dots, z^{k_m})$ for integers k_1, \dots, k_m . Since $\det P_0(z) = z^n$ with $n = \sum_{j=1}^m n_j$, the sum of exponents of z on the diagonal of P_0 , we have $\det D_0(z) = z^n$ so that $-\sum_{j=1}^m n_j = -n = -\sum_{j=1}^m k_j$ is the Fredholm index of T_{P_0} .

Since $\Omega_0 = \text{Diag}(\phi_1, \dots, \phi_m)$ and $z^{-k}I_m$ are diagonal matrices, the question on Fredholm properties of $T_{z^{-k}\Omega_0}$ viz-a-viz $T_{z^{-k}I_m} T_{\Omega_0}$, reduces to the scalar case for each diagonal entry $z^{-k}\phi_j$, $j = 1, \dots, m$. It follows from Proposition 2.4 that $T_{z^{-k}\Omega_0}$ is Fredholm if and only if all operators $T_{z^{-k}\phi_j}$, $j = 1, \dots, m$, are Fredholm, and in this case

$$\text{Index}(T_{z^{-k}\Omega_0}) = \sum_{j=1}^m \text{Index}(T_{z^{-k}\phi_j}).$$

We can thus invoke Theorem 1.1 from [11] to conclude that $T_{z^{-k}\Omega_0}$ is Fredholm, or equivalently, T_{Ω} is Fredholm, if and only if non of the entries ϕ_j of Ω_0 has a zero on \mathbb{T} (equivalently, $\det \Omega_0(z)$ has no zeroes on \mathbb{T}). Furthermore, in case $T_{z^{-k}\Omega_0}$ is Fredholm we have $\phi_j = 1/q_j$ for some $q_j \in \mathcal{P}$ and

$$\text{Index}(T_{z^{-k}\Omega_0}) = \sum_{j=1}^m k + \deg q_j = mk + \sum_{j=1}^m \deg q_j.$$

Combining the above observations we obtain that T_{Ω} is Fredholm if and only if $\det \Omega_0(z)$ has no zeroes on \mathbb{T} , and in case T_{Ω} is Fredholm we have

$$\text{Index}(T_{\Omega}) = \text{Index}(T_{z^{-k}I_m} T_{\Omega_0} T_{P_0}) = \text{Index}(T_{z^{-k}I_m} T_{\Omega_0}) + \text{Index}(T_{P_0})$$

$$= mk + \sum_{j=1}^m \deg q_j - \sum_{j=1}^m n_j,$$

where $n_1, \dots, n_m \geq 0$ are the exponents of z on the diagonal entries of P_0 and q_1, \dots, q_m are the denominators of the co-prime representations of the diagonal entries of Ω_o . This completes the proof of Theorem 1.4. \square

Remark 7.1 It now follows from the decomposition of $\Omega(z) = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix}$ in Sect. 5 that T_Ω is not Fredholm, despite $\det \Omega(z)$ not having a zero on \mathbb{T} .

From the proof of Theorem 1.4 it follows that T_Ω is Fredholm if and only if T_Ξ is Fredholm, where

$$\Xi(z) = z^{-k} \Omega_o(z) P_0(z)$$

with k , Ω_o and P_0 from any Wiener–Hopf type factorization of Ω as in Theorem 1.1. Moreover, if T_Ω is Fredholm, then $\text{Index}(T_\Omega) = \text{Index}(T_\Xi)$, in fact, one has $\dim \text{Ker } T_\Omega = \dim \text{Ker } T_\Xi$ and $\text{codim Ran } T_\Omega = \text{codim Ran } T_\Xi$. Assume T_Ω is Fredholm, so that Ω_o has the form $\Omega_o(z) = \text{Diag}(1/q_1, \dots, 1/q_m)$ with q_j a divisor of q_{j+1} for $j = 1, \dots, m-1$. Let the diagonal entries of P_0 be z^{n_1}, \dots, z^{n_m} and let $p_{i,j} \in \mathcal{P}$ be the lower triangular off-diagonal entry in position (i, j) , for $i > j$. By construction $\deg p_{i,j} < n_i$. Then Ξ has the form

$$\Xi(z) = \begin{pmatrix} \frac{z^{n_1}}{z^k q_1(z)} & 0 & \cdots & 0 \\ \frac{p_{2,1}(z)}{z^k q_2(z)} & \frac{z^{n_2}}{z^k q_2(z)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{p_{m,1}(z)}{z^k q_m(z)} & \cdots & \frac{p_{m,m-1}(z)}{z^k q_m(z)} & \frac{z^{n_m}}{z^k q_m(z)} \end{pmatrix}. \quad (7.1)$$

One may wonder to what extent this form is unique. For instance, is it maybe the case that the numbers n_1, \dots, n_m are unique? The following examples show that this is not the case.

Example 7.2 Let

$$\Xi(z) = \begin{pmatrix} \frac{1}{(z-1)^3} & 0 \\ \frac{z^2}{(z-1)^4} & \frac{z^5}{(z-1)^4} \end{pmatrix}.$$

This function is of the form $\Xi_0(z) P_0(z)$, with $\Xi_0(z) = \text{diag} \left(\frac{1}{(z-1)^3}, \frac{1}{(z-1)^4} \right)$ and $P_0(z) = \begin{pmatrix} 1 & 0 \\ z^2 & z^5 \end{pmatrix}$. So for this factorization we have that $n_1 = 0$, $n_2 = 5$, and $k = 0$.

Introduce

$$\Xi_-(z) = \begin{pmatrix} -1 & \frac{z-1}{z^2} \\ 0 & 1 \end{pmatrix}, \quad \Xi_+(z) = \begin{pmatrix} -z^3 & 1 \\ 1 & 0 \end{pmatrix},$$

then Ξ_- and its inverse are minus functions and Ξ_+ and its inverse are plus functions, and

$$\Xi(z) = \Xi_-(z)^{-1} \begin{pmatrix} \frac{z^3}{(z-1)^3} & 0 \\ 0 & \frac{z^2}{(z-1)^4} \end{pmatrix} \Xi_+(z)^{-1},$$

which is more like an ordinary Wiener–Hopf factorization. Note that the middle factor is also of the form as in Theorem 1.2, but now with $n_1 = 3$ and $n_2 = 2$. In line with Theorem 1.2, the denominators on the diagonal, $(z-1)^3$ and $(z-1)^4$, do not change.

Both factorizations tell us that T_Ξ is Fredholm with index 2. It is obvious we would prefer the second factorization, as in that factorization the degrees of q_1 and q_2 and the n_1 and n_2 give us information on the dimension of the kernel and codimension of the range of T_Ξ using Proposition 2.4. In fact, it can be checked directly that the dimension of the kernel of T_Ξ is two and since T_Ξ is Fredholm with index two it follows that T_Ξ is onto.

The example raises the question whether it might always be possible to diagonalize the middle term (7.1) by multiplying on the left with a minus function that has a minus function inverse and on the right with a plus function that has a plus function inverse. The following example shows a more general 2×2 case, where the procedure from Example 7.2 can be carried out.

Example 7.3 Start by considering

$$\Omega(z) = \begin{pmatrix} \frac{z^{k_1}}{q_1(z)} & 0 \\ \frac{d(z)}{q_2(z)} & \frac{z^{k_2}}{q_2(z)} \end{pmatrix}$$

where q_1 a divisor of q_2 and q_2 has all its roots on \mathbb{T} , and $\deg d < k_2$. Write $d(z) = z^{k_{12}}d_0(z)$, with $d_0(0) \neq 0$, and $k_{12} + \deg d_0 < k_2$. Then $d_0(z)$ and $z^{k_2-k_{12}}$ have greatest common divisor 1, and so by the Bezout identity there are polynomials $p_1(z)$ and $p_2(z)$ such that $d_0(z)p_1(z) + z^{k_2-k_{12}}p_2(z) = 1$, and the degree of $p_1(z)$ is less than or equal to $k_2 - k_{12} - 1$, while the degree of $p_2(z)$ is less than or equal to the degree of d_0 minus one. Set

$$\Omega_+(z) = \begin{pmatrix} -z^{k_2-k_{12}} & p_1(z) \\ d_0(z) & p_2(z) \end{pmatrix}.$$

Then Ω_+ is a plus function, and since the determinant of Ω_+ is one by the Bezout identity, also the inverse of Ω_+ is a plus function. Now

$$\Omega(z)\Omega_+(z) = \begin{pmatrix} -\frac{z^{k_1+k_2-k_{12}}}{q_1(z)} & \frac{z^{k_1}p_1(z)}{q_1(z)} \\ 0 & \frac{z^{k_{12}}}{q_2(z)} \end{pmatrix}.$$

Write $q_2(z) = q_0(z)q_1(z)$. Now assume that

$$k_1 + \deg q_0 + \deg p_1 \leq k_{12}. \quad (7.2)$$

Set

$$\Omega_{-}(z) = \begin{pmatrix} -1 & \frac{z^{k_1} p_1(z) q_0(z)}{z^{k_{12}}} \\ 0 & 1 \end{pmatrix}$$

which is a minus function with an inverse that is also a minus function. Then

$$\Omega_{-}(z) \Omega(z) \Omega_{+}(z) = \begin{pmatrix} \frac{z^{k_1+k_2-k_{12}}}{q_1(z)} & 0 \\ 0 & \frac{z^{k_{12}}}{q_2(z)} \end{pmatrix}.$$

Note that in the previous example condition (7.2) is satisfied. Without (7.2), however, Ω_{-} above would have a pole at ∞ and hence would not be a minus function.

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